Int. J. Solids Structures Vol. 29, No. 16, pp. 2013–2021, 1992
Printed in Great Britain.
0020–7683/92 \$5.00 + .00

REFINED ANALYSIS OF ELASTIC–PLASTIC CRYSTALS

WIKTOR GAMBIN
Institute of Fundamental Technological Research, Polish Academy of Sciences, Swietokrzyska 21, 00-49 Warsaw, Poland

(*Received* 18 *July* 1991)

Abstract—The concept of the rigid–ideally plastic crystals with interacting slip systems is extended for the case of the elastic–plastic crystals with work hardening. It allows the crystal analysis to be performed in a similar way to the case of elastic–plastic continuum at large strains, i.e. on the basis of the complete system of equations. As in classical plasticity, the flow rule is expressed in terms of the Kirchhoff stress increment and the strain rate tensor. Three additional constitutive relations between the plastic spin and stress components help to describe the lattice rotations. A smooth but highly nonlinear yield condition is assumed as an approximation of the Schmid law. The hardening law describing the self-hardening and the latent hardening is considered within the isotropic and the kinematic approach. The complete model is applied to an analysis of the drawing process of f.c.c. single crystals. By comparison with the numerical procedure based on the conventional rate-independent approach, the calculations are simplified considerably.

INTRODUCTION

The rate-independent crystal plasticity formulated by Hill and Rice (1972) is based on the Schmid law as a yield condition. The existence of the plastic corners on the yield surfaces leads to an ambiguity in a behaviour of the model. The rate-dependent approach (Asaro and Needleman, 1985) overcomes the above difficulty. Such a formulation is satisfactory for the processes with a deformation rate close to an assumed *a priori* reference strain rate. For arbitrary processes, the model of crystals with interacting slip systems (Gambin, 1991a, b) has been proposed. The model is based on smooth yield surfaces with rounded corners, which can be arbitrarily close to those generated by the Schmid law. The smoothness of the yield surfaces ensures the uniqueness of a model behaviour. In particular, it is possible to formulate the complete system of equations for initial-boundary problems, including lattice reorientations caused by large plastic strains. The model has been worked out for rigid–ideally plastic crystals. An extension of the model for elastic–plastic work hardening problems is the subject of this paper.

EXISTING MODELS OF ELASTIC–PLASTIC CRYSTALS

Because the kinematics of elastic–plastic crystals is well described in the literature [cf. Asaro (1983)], let us focus our considerations on the constitutive relations. All quantities will be referred to a fixed Cartesian system of coordinates. Let $\tau_{ij} = (\rho/\rho_0)\sigma_{ij}$ be the Kirchhoff stress tensor, where σ_{ij} is the Cauchy stress, and ρ_0, as well as ρ, are the mass densities in the reference and current configurations, respectively. A stress increment is described by the Zaremba–Jaumann derivative of the Kirchhoff stress:

$$\overset{\nabla}{\tau}_{ij} = \dot{\tau}_{ij} - \omega_{ik}\tau_{kj} + \tau_{ik}\omega_{kj} \tag{1}$$

where $\dot{\tau}_{ij}$ is the material derivative of τ_{ij}, and ω_{ij} is the total material spin. The general form of the constitutive law for elastic–plastic crystals is as follows [see rule (3.17) in Asaro (1983)]:

$$\tau_{ij}^{\nabla} = \mathscr{L}_{ijkl}d_{kl} - \sum_{\alpha=1}^{M} [\mathscr{L}_{ijkl}P_{kl}^{*(\alpha)} + \beta_{ij}^{(\alpha)}]\dot{\gamma}^{(\alpha)}. \quad (2)$$

In rule (2), $\mathscr{L}_{ijkl}$ are elastic moduli, d_{ij} is the total strain rate, and $\dot{\gamma}^{(\alpha)}$ are slip rates on slip systems: $\alpha = 1, 2, \ldots, M$. Moreover,

$$\beta_{ij}^{(\alpha)} = W_{ik}^{*(\alpha)}\tau_{kj} - \tau_{ik}W_{kj}^{*(\alpha)}. \quad (3)$$

The functions

$$P_{ij}^{*(\alpha)} = \tfrac{1}{2}(s_i^{*(\alpha)}m_j^{*(\alpha)} + m_i^{*(\alpha)}s_j^{*(\alpha)}) \quad \text{and} \quad W_{ij}^{*(\alpha)} = \tfrac{1}{2}(s_i^{*(\alpha)}m_j^{*(\alpha)} - m_i^{*(\alpha)}s_j^{*(\alpha)}) \quad (4, 5)$$

are defined by the current orientation of the slip direction $s_i^{*(\alpha)}$ and the normal to the slip plane $m_i^{*(\alpha)}$. Denote by $s_i^{(\alpha)}$ and $m_i^{(\alpha)}$ the slip direction and the normal to the slip plane in the reference configuration. It is assumed that

$$s_i^{*(\alpha)} = F_{ij}^{*}s_j^{(\alpha)} \quad \text{and} \quad m_j^{*(\alpha)} = m_i^{(\alpha)}F_{ij}^{*-1} \quad (6, 7)$$

where F_{ij}^{*} is the elastic part of the total deformation gradient F_{ij}. Recall that

$$F_{ij} = F_{ik}^{*}F_{kj}^{P} \quad (8)$$

where F_{ij}^{*} describes a stretching and rotation of the crystalline lattice, and F_{ij}^{P} is the plastic part of F_{ij} due solely to slip.

Note that introducing the plastic strain rate

$$d_{ij}^{P} = \sum_{\alpha=1}^{M} P_{ij}^{*(\alpha)}\dot{\gamma}^{(\alpha)} \quad (9)$$

and the plastic spin

$$\omega_{ij}^{P} = \sum_{\alpha=1}^{M} W_{ij}^{*(\alpha)}\dot{\gamma}^{(\alpha)} \quad (10)$$

one can write relation (2) in the following alternative form:

$$\tau_{ij}^{\nabla} = \mathscr{L}_{ijkl}d_{kl} - \sum_{\alpha=1}^{M} (\mathscr{L}_{ijkl}d_{kl}^{P} + \omega_{ik}^{P}\tau_{kj} - \tau_{ik}\omega_{kj}^{P}). \quad (11)$$

As in the case of classical continuum plasticity we are looking for a relation between τ_{ij}^{∇} and d_{ij}. To obtain it, one can express the slip rates $\dot{\gamma}^{(\alpha)}$ in (2) in terms of the total strain rate d_{kl}, as in the rate-independent model introduced by Hill and Rice (1972). The other way is to express the slip rates $\dot{\gamma}^{(\alpha)}$ by the stress tensor τ_{ij}, as in the rate-dependent model proposed by Asaro and Needleman (1985). Recall the above formulations.

Rate-independent model

Here, the demanded relation between the slip rates and the total strain rate is the following [see Asaro (1983)]:

$$\dot{\gamma}^{(\alpha)} = \sum_{\alpha=1}^{M} g_{\alpha\beta}^{-1}\lambda_{ij}^{(\beta)}d_{ij} \quad \text{where} \quad \lambda_{ij}^{(\beta)} = \mathscr{L}_{ijkl}P_{kl}^{*(\beta)} + \beta_{ij}^{(\beta)} \quad (12, 13)$$

and

$$g_{\alpha\beta} = h_{\alpha\beta} + \lambda_{ij}^{(\alpha)} P_{ij}^{*(\beta)}. \tag{14}$$

The hardening moduli $h_{\alpha\beta}$ determine an increment in the current critical shear stress

$$\dot{\tau}_c^{(\alpha)} = \sum_{\beta=1}^{M} h_{\alpha\beta}\dot{\gamma}^{(\beta)} \quad \text{for} \quad \dot{\gamma}^{(\beta)} \geqslant 0; \quad \beta = 1, 2, \ldots, M. \tag{15, 16}$$

In relation (12), the $\dot{\gamma}$s are uniquely determined if the matrix $g_{\alpha\beta}$ is positive definite. It is satisfied when the number of potentially active slip systems does not exceed five. In this case the matrix $g_{\alpha\beta}^{-1}$ exists and one can introduce (12) into (2) in order to obtain the final form of the constitutive relation

$$\tau_{ij}^{\nabla} = \mathscr{C}_{ijkl}\mathrm{d}_{kl} \quad \text{where} \quad \mathscr{C}_{ijkl} = \mathscr{L}_{ijkl} - \sum_{\alpha=1}^{M}\sum_{\beta=1}^{M} \lambda_{ij}^{(\alpha)} g_{\alpha\beta}^{-1} \lambda_{kl}^{(\beta)} \tag{17, 18}$$

are elastic–plastic moduli.

Now, the constitutive relation has the explicit form, but the choice of the active slip systems makes any analysis a very cumbersome one.

Rate-dependent model

To avoid the above difficulty, a plastic behaviour of crystals is modelled by a viscous one, according to the rule

$$\dot{\gamma}^{(\alpha)} = \dot{a}^{(\alpha)} \left[\frac{\tau^{(\alpha)}}{g^{(\alpha)}}\right] \left[\left|\frac{\tau^{(\alpha)}}{g^{(\alpha)}}\right|\right]^{(1/m)-1} \tag{19}$$

where $\dot{a}^{(\alpha)}$ is an assumed *a priori* reference strain rate. The functions $g^{(\alpha)}$ describe the crystal work hardening. Their evolution is specified by the hardening law

$$\dot{g}^{(\alpha)} = \sum_{\alpha=1}^{M} h_{\alpha\beta}|\dot{\gamma}^{(\beta)}| \tag{20}$$

where $h_{\alpha\beta}$ are hardening moduli. The component $1/m$ characterizes the material rate sensitivity, which diminishes when m is tending to zero. Introducing relation (19) into (2) one can obtain a viscous flow rule in which the stress increment τ_{ij}^{∇} is expressed by the strain rate d_{ij} and the stress tensor τ_{ij}.

In the model, the slip shear rates are uniquely specified by (19), and the problem of choice of active slip systems disappears. However, the model depends on the assumed *a priori* reference strain rate $\dot{a}^{(\alpha)}$.

THE PROPOSED MODEL

The model of rigid–ideally plastic crystals which uniquely describes the really plastic (rate-independent) crystal behaviour has been given by Gambin (1991a) [see also Gambin (1991b)]. The model may be extended in the case of elastic–plastic work hardened crystals. Below, two approaches to the work hardening description are presented.

Isotropic approach

The *slip systems interaction rule* proposed by Gambin (1991a) leads to the relation

$$\dot{\gamma}^{(\alpha)} = \frac{\lambda}{\tau_c^{(\alpha)}}\left(\frac{\tau^{(\alpha)}}{\tau_c^{(\alpha)}}\right)^{2n-1} \tag{21}$$

where an increment of the current critical shear stress $\tau_c^{(\alpha)}$ is prescribed by the hardening law

$$\dot{\tau}_c^{(\alpha)} = \sum_{\alpha=1}^{M} h_{\alpha\beta}|\dot{\gamma}^{(\beta)}|. \tag{22}$$

The parameter $n = 1, 2, \ldots,$ in the exponent of the relation (21) is assumed to be a material constant and determines a degree of interactions between slip systems.

Rule (21) is very similar to relation (19) which describes the viscous crystals, but instead of the assumed *a priori* reference strain rate $\dot{a}^{(\alpha)}$, a non-negative function of the loading process λ is introduced. This function is calculated from the following smooth yield condition :

$$\sum_{\alpha=1}^{M}\left(\frac{\tau^{(\alpha)}}{\tau_c^{(\alpha)}}\right)^{2n} - \frac{1}{M}\sum_{\alpha=1}^{M}\sum_{\beta=1}^{M}\left(2\frac{\tau^{(\beta)}}{\tau_c^{(\alpha)}}P_{ij}^{(\beta)}P_{ij}^{(\alpha)}\right)^{2n} = 0 \tag{23}$$

where

$$P_{ij}^{(\alpha)} = \tfrac{1}{2}(s_i^{(\alpha)}m_j^{(\alpha)} + m_i^{(\alpha)}s_j^{(\alpha)}) \tag{24}$$

are the functions $P_{ij}^{*(\alpha)}$ related to the reference configuration.

The above yield condition, associated with the *slip system interaction rule*, is the crucial point of the presented formulation. For $n = 1$, it gives the Mises criterion with quadratic yield surfaces. Generally, yield surfaces described by (23) have rounded-off plastic corners with a radius of curvature arbitrarily small, for large n. For $n = 15$, the yield surfaces generated by (23) approximate well with those created by the Schmid law.

To determine the function λ, denote by f the left side of the plasticity condition (23). Then the consistency condition : $\dot{f} = 0$ takes the following form :

$$\sum_{\alpha=1}^{M}\left(\frac{\dot{\tau}^{(\alpha)}}{\tau_c^{(\alpha)}} - \frac{\dot{\tau}_c^{(\alpha)}}{\tau_c^{(\alpha)}}\right)\left(\frac{\tau^{(\alpha)}}{\tau_c^{(\alpha)}}\right)^{2n} = \frac{1}{M}\sum_{\alpha=1}^{M}\sum_{\beta=1}^{M}\left(\frac{\dot{\tau}_c^{(\beta)}}{\tau_c^{(\beta)}} - \frac{\dot{\tau}_c^{(\alpha)}}{\tau_c^{(\alpha)}}\right)\left(2\frac{\tau^{(\beta)}}{\tau_c^{(\alpha)}}P_{ij}^{(\beta)}P_{ij}^{(\alpha)}\right)^{2n}. \tag{25}$$

One can prove [see (2.56) and (2.59) in Asaro (1983)], that

$$\dot{\tau}^{(\alpha)} = \lambda_{ij}^{(\alpha)} d_{ij}^{*} \tag{26}$$

where $d_{ij}^{*} = d_{ij} - d_{ij}^{p}$ is the elastic part of the total strain rate tensor. Taking into account (9) and (21), the relation (26) may be written in the form

$$\dot{\tau}^{(\alpha)} = \lambda_{ij}^{(\alpha)} d_{ij} - \lambda\sum_{\beta=1}^{M}\frac{\lambda_{ij}^{(\alpha)}P_{ij}^{*(\beta)}}{\tau_c^{(\beta)}}\left(\frac{\tau^{(\beta)}}{\tau_c^{(\beta)}}\right)^{2n-1}. \tag{27}$$

On the other hand, by introducing (21) into (22) one can obtain

$$\dot{\tau}_c^{(\alpha)} = \lambda \sum_{\alpha=1}^{M} h_{\alpha\beta} \frac{1}{\tau_c^{(\beta)}} \left(\frac{|\tau^{(\alpha)}|}{\tau_c^{(\beta)}} \right)^{2n-1} \tag{28}$$

The relations (25), (27) and (28) enable the function λ to be expressed in terms of the total strain rate tensor

$$\lambda = \frac{\mathcal{F}_{ij}}{\mathcal{F}_{mn}\mathcal{G}_{mn} + h_0} d_{ij}. \tag{29}$$

In the above

$$\mathcal{F}_{ij} = \sum_{\alpha=1}^{M} \frac{\lambda_{ij}^{(\alpha)}}{\tau_c^{(\alpha)}} \left(\frac{\tau^{(\alpha)}}{\tau_c^{(\alpha)}} \right)^{2n-1} \qquad \mathcal{G}_{ij} = \sum_{\alpha=1}^{M} \frac{P_{ij}^{*(\alpha)}}{\tau_c^{(\alpha)}} \left(\frac{\tau^{(\alpha)}}{\tau_c^{(\alpha)}} \right)^{2n-1} \tag{30, 31}$$

$$h_0 = \sum_{\alpha=1}^{M} \sum_{\beta=1}^{M} \frac{h_{\alpha\beta}}{\tau_c^{(\alpha)}\tau_c^{(\beta)}} \left(\frac{\tau_c^{(\alpha)}}{\tau_c^{(\alpha)}} \right)^{2n} \left(\frac{|\tau^{(\beta)}|}{\tau_c^{(\beta)}} \right)^{2n-1}$$
$$+ \frac{1}{M} \sum_{\alpha=1}^{M} \sum_{\beta=1}^{M} \sum_{\delta=1}^{M} \left(\frac{h_{\beta\delta}}{\tau_c^{(\beta)}\tau_c^{(\delta)}} - \frac{h_{\alpha\delta}}{\tau_c^{(\alpha)}\tau_c^{(\delta)}} \right) \left(2 \frac{\tau_c^{(\beta)}}{\tau_c^{(\alpha)}} P_{ij}^{(\beta)} P_{ij}^{(\alpha)} \right)^{2n} \left(\frac{|\tau^{(\delta)}|}{\tau_c^{(\delta)}} \right)^{2n-1}. \tag{32}$$

Concluding, the incremental constitutive relation one can write in the following final form :

$$\tau_{ij}^{\nabla} = \left(\mathcal{L}_{ijkl} - \frac{\mathcal{F}_{ij}\mathcal{F}_{kl}}{\mathcal{F}_{mn}\mathcal{G}_{mn} + h_0} \right) d_{kl}. \tag{33}$$

Note that the Prandtl–Reuss equations have the same form in the case of isotropic plasticity at large strains [see rule (20) in McMeeking and Rice (1975)]. In that case it is enough to assume

$$\mathcal{F}_{ij} = \mathcal{L}_{ijkl}\sigma_{kl} \qquad \mathcal{G}_{ij} = \sigma'_{ij} \qquad h_0 = \tfrac{2}{3} h \sigma'_{mn}\sigma'_{mn} \tag{34, 35, 36}$$

where $\mathcal{L}_{ijkl}$ are the elastic moduli, σ'_{ij} is the Cauchy stress deviator, and h is the isotropic hardening modulus. The above has a great importance: exchanging rules (34)–(36) on (30)–(32), one can use the standard FEM procedures for a finite element analysis of elastic–plastic crystals.

Kinematic approach

Sometimes, it is convenient to assume a kinematic hardening law instead of (22). In this case, the incremental constitutive relation (33) is valid, provided that we redefine the functions $\mathcal{F}_{ij}$, $\mathcal{G}_{ij}$ and h_0. For this reason, consider the following relation:

$$\dot{\gamma}^{(\alpha)} = \frac{\lambda}{k_c^{(\alpha)}} \left(\frac{\tau^{(\alpha)} - a^{(\alpha)}}{k_c^{(\alpha)}} \right)^{2n-1} \tag{37}$$

instead of (21). In the above, $k_c^{(\alpha)}$ are the initial critical shear stresss, and $a^{(\alpha)}$ are the residual shear stresses for each of the slip systems prescribed by the hardening law

$$\dot{a}^{(\alpha)} = \sum_{\beta=1}^{M} h_{\alpha\beta}\dot{\gamma}^{(\beta)}. \tag{38}$$

The non-negative function λ in (37) is calculated from the following yield condition [compare this with (23)]:

$$\sum_{\alpha=1}^{M}\left(\frac{\tau^{(\alpha)}-a^{(\alpha)}}{k_c^{(\alpha)}}\right)^{2n}-\frac{1}{M}\sum_{\alpha=1}^{M}\sum_{\beta=1}^{M}\left(2\frac{k_c^{(\beta)}}{k_c^{(\alpha)}}P_{ij}^{(\beta)}P_{ij}^{(\alpha)}\right)^{2n}=0. \tag{39}$$

The initial yield surfaces described by (39) may be regarded as an approximation of those generated by the Schmid law. As previously, one can determine the function λ by letting f represent the left side of (39). Then, the consistency condition: $\dot{f}=0$ takes the form

$$\sum_{\alpha=1}^{M}\frac{\dot{\tau}^{(\alpha)}-\dot{a}^{(\alpha)}}{k_c^{(\alpha)}}\left(\frac{\tau^{(\alpha)}-a^{(\alpha)}}{k_c^{(\alpha)}}\right)^{2n-1}=0. \tag{40}$$

Taking into account (9) and (37), the relation (26) may be written in the form

$$\dot{\tau}^{(\alpha)}=\lambda_{ij}^{(\alpha)}d_{ij}-\lambda\sum_{\beta=1}^{M}\frac{\lambda_{ij}^{(\alpha)}P_{ij}^{(\beta)}}{k_c^{(\alpha)}}\left(\frac{\tau^{(\beta)}-a^{(\beta)}}{k_c^{(\beta)}}\right)^{2n-1}. \tag{41}$$

On the other hand, introducing (37) into (38) one can obtain

$$\dot{a}^{(\alpha)}=\lambda\sum_{\beta=1}^{M}h_{\alpha\beta}\frac{1}{k_c^{(\beta)}}\left(\frac{\tau^{(\beta)}-a^{(\beta)}}{k_c^{(\beta)}}\right)^{2n-1}. \tag{42}$$

The relations (40)–(42) enable the function λ to be expressed in terms of the total strain rate tensor

$$\lambda=\frac{\mathcal{F}_{ij}}{\mathcal{F}_{mn}\mathcal{G}_{mn}+h_0}d_{ij}. \tag{43}$$

In the above,

$$\mathcal{F}_{ij}=\sum_{\alpha=1}^{M}\frac{\lambda_{ij}^{(\alpha)}}{k_c^{(\alpha)}}\left(\frac{\tau^{(\alpha)}-a^{(\alpha)}}{k_c^{(\alpha)}}\right)^{2n-1}\qquad \mathcal{G}_{ij}=\sum_{\alpha=1}^{M}\frac{P_{ij}^{*(\alpha)}}{k_c^{(\alpha)}}\left(\frac{\tau^{(\alpha)}-a^{(\alpha)}}{k_c^{(\alpha)}}\right)^{2n-1} \tag{44, 45}$$

$$h_0=\sum_{\alpha=1}^{M}\sum_{\beta=1}^{M}\frac{h_{\alpha\beta}}{k_c^{(\alpha)}k_c^{(\beta)}}\left(\frac{\tau^{(\alpha)}-a^{(\alpha)}}{k_c^{(\alpha)}}\right)^{2n-1}\left(\frac{\tau^{(\beta)}-a^{(\beta)}}{k_c^{(\beta)}}\right)^{2n-1}. \tag{46}$$

Then, the incremental constitutive relation takes the form of the rule (33). As in the isotropic approach, one can adopt the standard FEM procedures for a finite element analysis of elastic–plastic crystals. Note that the kinematic approach gives some advantages: one can determine values of residual stresses in a crystal, and the expression for the kinematic hardening function h_0, given by (46) is simpler than for the isotropic one [see rule (32)].

LATTICE REORIENTATIONS

An analysis of elastic–plastic crystals at small strains is quite simple: for an initial lattice orientation, one should calculate the functions $P_{ij}^{(\alpha)}$ according to (24), to check the plasticity condition (23) or (39) and to determine the functions $\mathcal{F}_{ij}$, $\mathcal{G}_{ij}$ and h_0 according to the rules (29)–(32) or (44)–(46).

In the case of large plastic strains, the plasticity condition and the functions $\mathcal{F}_{ij}$, $\mathcal{G}_{ij}$ and h_0 are determined by the quantities $P_{ij}^{*(\alpha)}$ [see definition (4)]. These quantities are related to the current lattice configuration described by

$$F^*_{ij} = U^*_{ik}R^*_{kj} \tag{47}$$

where U^*_{ik} represents the elastic stretching, and R^*_{kj} represents the lattice rotation with respect to the reference configuration. Neglecting an influence of the lattice stretching on the slip system vectors, one can assume

$$s_i^{*(\alpha)} = R^*_{ij}s_j^{(\alpha)} \quad \text{and} \quad m_{ij}^{*(\alpha)} = R^*_{ij}m_j^{(\alpha)} \tag{48, 49}$$

instead of (6)–(7). An increment of R^*_{ij} may be found from the following rule:

$$\dot{R}^*_{ij} = \omega^*_{ik}\,R^*_{kj} \tag{50}$$

where the lattice spin ω^*_{ij} is the difference between the total spin ω_{ij} and its plastic part ω^{P}_{ij}. Introducing (21) and (29)–(32) [or (37) and (43)–(46)] into (10), one can obtain

$$\omega^{\mathrm{P}}_{ij} = \frac{\mathscr{F}_{kl}d_{kl}}{\mathscr{F}_{mn}\mathscr{G}_{mn}+h_0}\mathscr{H}_{ij}. \tag{51}$$

For $W_{ij}^{*(\alpha)}$ prescribed by (5), the function $\mathscr{H}_{ij}$ within the isotropic approach is defined as follows

$$\mathscr{H}_{ij} = \sum_{\alpha=1}^{M}\frac{W_{ij}^{*(\alpha)}}{\tau_c^{(\alpha)}}\left(\frac{\tau^{(\alpha)}}{\tau_c^{(\alpha)}}\right)^{2n-1} \tag{52}$$

and within the kinematic approach—by the rule

$$\mathscr{H}_{ij} = \sum_{\alpha=1}^{M}\frac{W_{ij}^{*(\alpha)}}{k_c^{(\alpha)}}\left(\frac{\tau^{(\alpha)}-a^{(\alpha)}}{k_c^{(\alpha)}}\right)^{2n-1}. \tag{53}$$

Then, for a given velocity field, v_i,

$$\omega^*_{ij} = \tfrac{1}{2}(v_{i,j}-v_{j,i}) - \frac{\mathscr{F}_{kl}d_{kl}}{\mathscr{F}_{mn}\mathscr{G}_{mn}+h_0}\mathscr{H}_{ij}. \tag{54}$$

An incremental procedure based on rule (50) does not ensure the orthogonality of the updated rotation matrix R^*_{ij}. For this reason, it is better to express R^*_{ij} in terms of three Euler angles $\{\varphi_1, \Phi, \varphi_2\}$, and next to calculate their increments $\{\dot{\varphi}_1, \dot{\Phi}, \dot{\varphi}_2\}$ from the following rules [cf. Clement and Coulomb (1979)]:

$$\dot{\varphi}_1 = -\frac{\sin\varphi_1}{\sin\Phi}\omega^*_{23}+\frac{\cos\varphi_2}{\sin\Phi}\omega^*_{13} \quad \dot{\Phi} = -\cos\varphi_2\omega^*_{23}-\sin\varphi_2\omega^*_{13} \quad \dot{\varphi}_2 = -\dot{\varphi}_1\cos\Phi-\omega^*_{12}. \tag{55, 56, 57}$$

In the literature, the above rules are usually given in terms of the rotation vector r_i connected with the lattice spin by the relation

$$r_i = \tfrac{1}{2}\varepsilon_{ijk}\omega^*_{kj} \tag{58}$$

where ε_{ijk} is the antisymmetric permutational symbol.

LATENT HARDENING

The hardening moduli $h_{\alpha\beta}$ introduced in (22) and (38) are known in the literature as "slip-plane hardening rates" (Asaro, 1983). The diagonal terms of the matrix $h_{\alpha\beta}$ represent the "self-hardening" on the considered slip system, whereas the off-diagonal terms describe the "latent hardening" on the remains slip systems. A ratio of the latent hardening to the self-hardening takes values from the interval $1 \leqslant q \leqslant 1.4$. Because this ratio is larger than one, the lattice rotations "overshoot" the symmetry position between two conjugate slip systems. The above effect is commonly reported in the literature (Asaro, 1983).

Let us investigate the phenomenon with the example of the drawing process of single f.c.c. crystals. Consider reorientations of longitudinal axes of extended crystalline samples to the fixed crystallographic directions during extension reaching up to 100%. Assume, within the kinematic approach, that the hardening moduli are prescribed by the rule

$$h_{\alpha\beta} = h[q\mathbf{I}_{\alpha\beta} + (1-q)\delta_{\alpha\beta}] \tag{59}$$

where, $\mathbf{I}_{\alpha\beta}$ is the matrix with all elements equal to one, $\delta_{\alpha\beta}$—Kronecker symbol, h—a constant self-hardening rate (the same for all slip systems), and q—the latent hardening ratio. The results of calculations for seven crystals with various initial orientations and for the parameters: $n = 15$, $h = 0.01$ and $q = 1.1$ are shown in Fig. 1. In every considered case, one can observe the overshoot phenomenon. It is interesting to compare the above results with those obtained for the rigid–ideally plastic crystals [see Fig. 1 in Gambin (1991a)].

Last of all, the simplicity of the performed analysis should be highlighted: all calculations were executed with the aid of the table calculator H-P 9830A with internal memory 4.5 kB.

CONCLUDING REMARKS

The concept of elastic–plastic crystals with interacted slip systems helps to formulate the problem of the crystal analysis on the base of the complete system of equations, as it was done in classical continuum plasticity. The obtained constitutive relation has the same form as that derived by McMeeking and Rice (1975) for the isotropic plasticity at large strains. Working on the basis of the introduced model, one can adopt the standard FEM procedures for a numerical analysis of elastic–plastic crystals. The proposed model may be a starting-point for an analysis of elastic–plastic polycrystals.

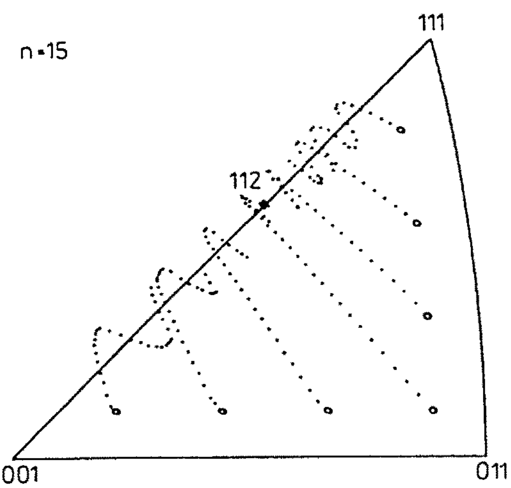

Fig. 1. Lattice reorientations during the extension of f.c.c. crystals with maximum elongation of 100%. The distance between successive dots corresponds to a strain of 2.5%.

REFERENCES

Asaro, R. J. (1983). Crystal plasticity. *ASME J. Appl. Mech.* **50**, 921–934.

Asaro, R. J. and Needleman, A. (1985). Texture development and strain hardening in rate dependent polycrystals. *Acta Metall.* **33**(6), 923–953.

Clement, A. and Coulomb, P. (1979). Eulerian simulation of deformation textures. *Scripta Metall.* **13**, 899–901.

Gambin, W. (1991a). Crystal plasticity based on yield surfaces with rounded-off corners. *ZAMM* **71**(4), T 265–T 268.

Gambin, W. (1991b). Plasticity of crystals with interacting slip systems. *Engng Trans.* **39**(3/4) (in press).

Hill, R. and Rice, J. R. (1972). Constitutive analysis of elastic–plastic crystals at arbitrary strain. *J. Mech. Phys. Solids* **20**, 401–413.

McMeeking, R. M. and Rice, J. R. (1975). Finite-element formulation for problems of large elastic–plastic deformation. *Int. J. Solids Structures* **11**, 601–616.